Re-Exam

Statistical Physics
2021-2022

Thursday, April 14, 2022, 15:00-17:00

Read these instructions carefully before making the exam!

- **Write your name and student number on *every* sheet.**
- ***Make sure to write readable for other people than yourself. Points will NOT be given for answers in illegible writing.***
- ***Language*; your answers have to be in English.**
- **Use of a (graphing) calculator and a cheat-sheet (single-sided A4) is allowed.**
- **This exam consists of 4 problems.**
- **The weight of the problems is Problem 1 (P1=25 pts); Problem 2 (P2=25 pts); Problem 3 (P3=25 pts) and Problem 4 (P4=25 pts). Weights of the various subproblems are indicated at the beginning of each problem.**
- **The grade of the exam is calculated as (P1+P2+P3 +P4)/10.**
- **For all problems you have to write down your arguments and the intermediate steps in your calculation, *else the answer will be considered as incomplete and points will be deducted.***

PROBLEM 1

Score: a+b+c+d =6+6+7+6 =25

According to quantum theory, the energy levels of a harmonic oscillator are given by $\varepsilon_n = \hbar\omega\left(n+\frac{1}{2}\right)$; $n = 0,1,2\cdots$, with ω the angular frequency of the oscillator. Assume that this oscillator is in equilibrium with a heat bath at temperature T.

a) Show that the partition function of the oscillator is given by,

$$Z_1 = \frac{1}{2\sinh\frac{x}{2}} \quad \text{with } x = \frac{\hbar\omega}{kT}$$

The mean energy $\langle\varepsilon\rangle$ of the oscillator is given by:

$$\langle\varepsilon\rangle = \frac{1}{2}\hbar\omega\frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}}$$

Now consider an 1-dimensional solid that consists of N atoms. Assume that the solid can be described as a system of N independent oscillators each with the same angular frequency ω (thus, use Einstein's theory!).

b) Show that the heat capacity C_V of this 1-dimensional solid can be written as:

$$C_V = Nk\left(\frac{x}{2}\right)^2\left(\frac{1}{\sinh^2\frac{x}{2}}\right)$$

c)

Sketch a graph of C_V as a function of $\frac{kT}{\hbar\omega}\left(=\frac{1}{x}\right)$. Justify the values you use in your sketch for $C_V\left(\frac{kT}{\hbar\omega}=0\right)$ and $C_V\left(\frac{kT}{\hbar\omega}=\infty\right)$.

The Hamiltonian H of the 1-dimensional oscillator is,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

with m the mass of the oscillating atom and p and q the momentum and the position coordinate of the atom.

d) Show that the heat capacity that is predicted from this Hamiltonian and the equipartition theorem agrees with the high temperature limit of the heat capacity that follows from the result of subproblem c).

PROBLEM 2

Score: a+b+c =8+9+8 =25

The grand partition function for a general system in equilibrium with a large reservoir with temperature T and chemical potential μ is given by,

$$\mathcal{Z} = \sum_{N=0}^{\infty} \sum_{r} e^{\beta(\mu N - E_r(N))}$$

with N the number of particles of the system and $E_r(N)$ the energy if the system is in state r and it has N particles.

Consider a system with two energy levels. The system is in contact with both a heat bath at temperature T and a particle reservoir characterized by the chemical potential μ. Let N be the number of particles in the system and E the energy of the system. The system has the following states:

1) Unoccupied $N = 0; E = 0$
2) Occupied with energy 0; $N = 1, E = 0$
3) Occupied with energy ε; $N = 1, E = \varepsilon$

~~a)~~ Show that the grand partition function $\mathcal{Z}$ for this system can be written as:

$$\mathcal{Z} = 1 + a + ae^{-\beta\varepsilon} \text{ with } a = e^{\beta\mu}$$

~~b)~~ Show that the mean number of particles in this system is:

$$\langle N \rangle = \frac{a(1 + e^{-\beta\varepsilon})}{1 + a(1 + e^{-\beta\varepsilon})}$$

~~c)~~ Calculate the mean energy of this system if $\varepsilon = -1.5$ eV and the temperature is such that $kT = 3$ eV and $\mu = -1.0$ eV.

PROBLEM 3

Score: a+b+c =12+5+8=25

The Dieterici equation is an equation of state that is sometimes used to describe a real gas:

$$P(V-b) = RTe^{-\frac{a}{RTV}}$$

in which P, V, T are the pressure, the molar volume and the temperature of the gas, respectively. The constant a controls the attractive molecular interactions and the constant b corrects for the volume of the gas molecules.

a) Show that for a gas described by the Dieterici equation (a Dieterici gas), the critical temperature, pressure and volume are given by:

$$(T_c, P_c, V_c) = (\frac{a}{4Rb}, \frac{a}{4e^2b^2}, 2b)$$

The second coefficient of a Dieterici gas is given by:

$$B(T) = b - \frac{a}{RT}$$

b) Calculate the Boyle temperature T_b of the Dieterici gas. Why is the name Boyle temperature appropriate?

c) Use reduced coordinates to rewrite the Dieterici equation.

PROBLEM 4

Score: a+b+c+d =5+8+6+6=25

Consider a two-dimensional gas of non-interacting bosons that is trapped in a harmonic potential. The gas is in equilibrium with a heat bath at temperature T and a particle reservoir with chemical potential μ. The energy eigenvalues of such a trapped boson are $E(n_1, n_2) = E_0(n_1 + n_2)$ with $n_1 = 0,1,2,\cdots,\infty$ and $n_2 = 0,1,2,\cdots,\infty$.

a) Show that the number of states $g(E)dE$ in which the boson has an energy E between E and $E + dE$ is given by:

$$g(E)dE = \frac{E}{E_0^2}dE$$

Hint: the number of states $\Gamma(E)$ with energy smaller than E is given by:

$$\Gamma(E) = \frac{1}{2}\left(\frac{E}{E_0}\right)^2$$

b a) Show that the total number of bosons N can be written as:

$$N = N_0 + \frac{1}{E_0^2}\int_0^\infty \frac{E}{e^{\beta(E-\mu)} - 1}dE$$

where N_0 is the number of bosons in the ground state with $E = 0$. Give an expression for N_0 and explain why the ground state energy should be treated in this way.

c b) Give the definition of the critical temperature T_c for a boson gas and show that this critical temperature for the 2-D harmonically trapped boson gas is given by,

$$T_c = \frac{\sqrt{6N}E_0}{\pi k}$$

d c) Calculate the number of bosons in the ground state for $T < T_c$. Express your answer in terms of T, T_c and N.

Physical constants:

Avogadro's number:	$N_0 = 6.02 \times 10^{23}$ mol^{-1}
Planck's constant:	$h = 6.626 \times 10^{-34}$ Js
	$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}$ Js
Boltzmann's constant:	$k = 1.381 \times 10^{-23}$ J K^{-1}
Gas constant:	$R = 8.31$ J mol^{-1} K^{-1}
Speed of light:	$c = 3 \times 10^{8}$ m s^{-1}
Electron rest mass:	$m_e = 9.11 \times 10^{-31}$ kg
Proton rest mass:	$m_p = 1.67 \times 10^{-27}$ kg
Charge of the electron:	$e = 1.60 \times 10^{-19}$ C
Bohr magneton:	$\mu_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24}$ A m^2
Permeability of vacuum:	$\mu_0 = 4\pi \times 10^{-7}$ N A^{-2}
Molar volume at STP:	22.4 litre

Integrals:

n	$\int_0^\infty dx\, x^n e^{-ax^2} \quad (a > 0)$	$\int_0^\infty \frac{x^n dx}{e^x - 1}$	$\int_0^\infty \frac{x^n dx}{e^x + 1}$	$\int_0^\infty \frac{x^n e^x}{(e^x - 1)^2}$	$\int_0^\infty x^n \ln(1 - e^{-x}) dx$
0	$\frac{1}{2}\sqrt{\left(\frac{\pi}{a}\right)}$	diverges	ln 2	diverges	$-\frac{\pi^2}{6}$
1/2	$\frac{0.6127}{a^{3/4}}$	$2.612\frac{\sqrt{\pi}}{2}$	0.6781	diverges	$-1.341\frac{\sqrt{\pi}}{2}$
1	$\frac{1}{2a}$	$\frac{\pi^2}{6}$	$\frac{\pi^2}{12}$	diverges	-1.202
3/2	$\frac{0.4532}{a^{5/4}}$	$1.341\frac{3\sqrt{\pi}}{4}$	1.153		$-1.127\frac{3\sqrt{\pi}}{4}$
2	$\frac{1}{4a}\sqrt{\frac{\pi}{a}}$	2.404	1.803	$\frac{\pi^2}{3}$	$-\frac{\pi^4}{45}$
5/2	$\frac{1.662}{a^{7/4}}$	$1.127\frac{15\sqrt{\pi}}{8}$	3.083		-3.505
3	$\frac{1}{2a^2}$	$\frac{\pi^4}{15}$	$\frac{7\pi^4}{120}$	7.212	-6.221
7/2	$\frac{0.5665}{a^{9/4}}$	12.268	11.184		
4	$\frac{3\sqrt{\pi}}{8a^{5/2}}$	24.886	23.331	$\frac{4\pi^4}{15}$	

Also some integrals and formulas at the other side!

And some more integrals:

$\int_{-\infty}^{\infty} e^{-cx^2}dx = \sqrt{\frac{\pi}{c}}$	$\int_{-\infty}^{\infty} e^{-cx^4}dx = \frac{2\Gamma\left(\frac{5}{4}\right)}{c^{\frac{1}{4}}}$
$\int_{-\infty}^{\infty} x^2 e^{-cx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{c^3}}$	$\int_{-\infty}^{\infty} x^2 e^{-cx^4}dx = \frac{\Gamma\left(\frac{3}{4}\right)}{2c^{\frac{3}{4}}}$
$\int_{-\infty}^{\infty} x^4 e^{-cx^2}dx = \frac{3}{4}\sqrt{\frac{\pi}{c^5}}$	$\int_{-\infty}^{\infty} x^4 e^{-cx^4}dx = \frac{\Gamma\left(\frac{5}{4}\right)}{2c^{\frac{5}{4}}}$

Some more useful formulas

$$\sinh x = \frac{e^x - e^{-x}}{2}; \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Reciprocity theorem: $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$; Reciprocal theorem: $\left(\frac{\partial x}{\partial z}\right)_y = \left[\left(\frac{\partial z}{\partial x}\right)_y\right]^{-1}$

Maxwell's relations:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$
$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$$
$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$
$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p .$$